4S ACADEMY • DIGITAL SAT MATH • ADVANCED

800 Score Strategy: Hardest Geometry

The 12 geometry question types that decide top scores on Module 2

On the hard second module, the geometry that separates a 700 from an 800 is not exotic. It is a dozen core types, mastered so well that you name them on sight, avoid the built-in trap, and handle the hardest questions, which usually just combine two of them at once. Each type below gives you the **strategy**, a worked **problem**, its **trap**, and, where it helps, the **SAT move** on the digital test. The closing formula card flags which formulas the SAT gives you and which you must know.

Part 1. Circles

four of the hardest patterns

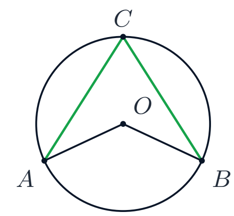
1. Central and inscribed angles

Strategy. An inscribed angle is half the central angle standing on the same arc. An angle inscribed in a semicircle is a right angle.

Problem. O is the centre and the central angle $\angle AOB = 108°$. Point C lies on the major arc. Find $\angle ACB$.

Solution. $\angle ACB$ and $\angle AOB$ stand on the same arc AB, so $\angle ACB = \frac{1}{2}(108°) = 54°$.

Trap. A point on the *minor* arc sees the major arc, giving $\frac{1}{2}(360 - 108) = 126°$. Always check which arc the vertex faces.



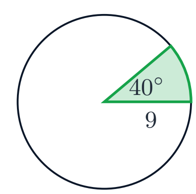
2. Arc length and sector area

Strategy. For a central angle θ in degrees, arc length $= \frac{\theta}{360} \cdot 2\pi r$ and sector area $= \frac{\theta}{360} \cdot \pi r^2$.

Problem. A circle has radius 9 and a sector with central angle $40°$. Find the arc length and the sector area, in terms of π.

Solution. Arc $= \frac{40}{360} \cdot 2\pi(9) = 2\pi$. Area $= \frac{40}{360} \cdot \pi(9)^2 = 9\pi$.

Trap. Arc uses $2\pi r$, area uses πr^2; do not swap them. If θ is in radians, use $r\theta$ and $\frac{1}{2}r^2\theta$ instead.



3. Equation of a circle

Strategy. $(x-h)^2+(y-k)^2=r^2$ has centre (h,k) and radius r. From a general form, complete the square in x and in y.

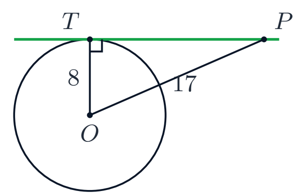
Problem. The equation $x^2+y^2-10x+6y+18=0$ defines a circle. Find its centre and radius.

Solution. $(x^2-10x)+(y^2+6y)=-18 \;\Rightarrow\; (x-5)^2-25+(y+3)^2-9=-18 \;\Rightarrow\; (x-5)^2+(y+3)^2=16$. Centre $(5,-3)$, radius 4.

Trap. The radius is $\sqrt{16}=4$, not 16. And the -25 and -9 created by completing the square must be carried to the right side.

SAT move. Type the equation into Desmos as it stands; it draws the circle, so you can read the centre and radius straight off the graph.

4. Tangent and radius

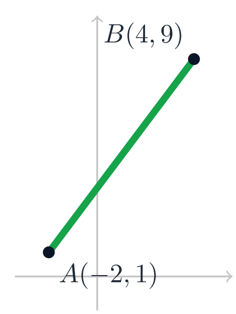


Strategy. A tangent meets the radius at a right angle at the point of contact. That right angle is almost always the key to a hidden right triangle.
Problem. From an external point P, a line is tangent to a circle at T. The centre is O, with $OT=8$ and $OP=17$. Find PT.
Solution. Since $OT \perp PT$, triangle OTP is right angled at T, so $PT=\sqrt{OP^2-OT^2}=\sqrt{17^2-8^2}=\sqrt{225}=15$.
Trap. The right angle is at T, not at P. OP is the hypotenuse, so subtract: never $\sqrt{OP^2+OT^2}$.

Part 2. Coordinate geometry

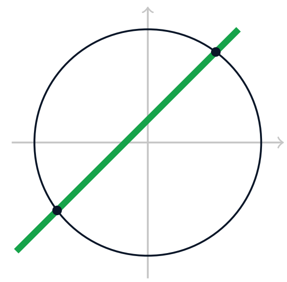
the plane, precisely

5. Distance, midpoint and slope



Strategy. Distance $=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$; midpoint is the average of the coordinates; parallel lines share a slope, perpendicular slopes multiply to -1.
Problem. $A(-2,1)$ and $B(4,9)$. Find the length of AB, its midpoint, and the slope of a line perpendicular to AB.
Solution. $AB=\sqrt{(4-(-2))^2+(9-1)^2}=\sqrt{36+64}=10$; midpoint $=\left(\frac{-2+4}{2},\frac{1+9}{2}\right)=(1,5)$. Slope of $AB=\frac{9-1}{4-(-2)}=\frac{4}{3}$, so a perpendicular has slope $-\frac{3}{4}$.
Trap. Mind the double negative: $4-(-2)=6$, not 2. A single sign slip here is the most common careless loss on the whole section.

6. Where a line meets a curve

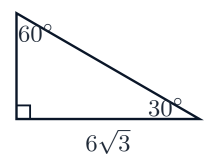


Strategy. To find where two graphs meet, substitute one into the other and solve, then pair each x with its y. A line meeting a circle is the classic hard version, combining this with type 3.
Problem. Where does the line $y = x+1$ meet the circle $x^2+y^2 = 25$?
Solution. Substitute $y = x+1$: $x^2+(x+1)^2 = 25 \Rightarrow 2x^2+2x-24 = 0 \Rightarrow x^2+x-12 = 0 \Rightarrow (x+4)(x-3) = 0$. So $x = -4$ or $x = 3$, giving $(-4,-3)$ and $(3,4)$.
Trap. A line can cut a circle at *two* points, so expect a quadratic and keep both solutions. Do not stop at one.
SAT move. The hardest Module 2 version gives the line an unknown and calls it *tangent*: tangent means exactly one solution, so set the quadratic's discriminant $b^2 - 4ac = 0$ and solve for the unknown. In Desmos, add a slider and slide until the line just touches.

Part 3. Triangles, trigonometry and solids

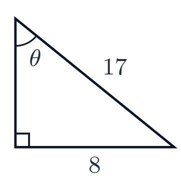
where most Module 2 marks are decided

7. Special right triangles



Strategy. Memorise both ratios. 45-45-90 is $1 : 1 : \sqrt{2}$; 30-60-90 is $1 : \sqrt{3} : 2$ (opposite $30°$, $60°$, $90°$). They collapse multi step problems into one step.
Problem. In a 30-60-90 triangle, the side opposite the $60°$ angle is $6\sqrt{3}$. Find the hypotenuse.
Solution. The side opposite $60°$ is $\sqrt{3}$ times the shortest side, so the shortest side is $\frac{6\sqrt{3}}{\sqrt{3}} = 6$. The hypotenuse is $2(6) = 12$.
Trap. The ratio is in order opposite $30°, 60°, 90°$. Assuming the given $6\sqrt{3}$ is the shortest side gives the wrong answer; match the side to its angle first.

8. Right triangle trigonometry



Strategy. $\sin =$ opp/hyp, $\cos =$ adj/hyp, $\tan =$ opp/adj. Label the sides relative to the named angle, then choose the ratio, filling the missing side with the Pythagorean theorem.
Problem. In a right triangle, the side opposite angle θ is 8 and the hypotenuse is 17. Find $\cos\theta$.
Solution. The adjacent side is $\sqrt{17^2 - 8^2} = 15$, so $\cos\theta =$ adj/hyp $= 15/17$.
Trap. Opposite and adjacent are relative to the angle you name. Here $\cos\theta = 15/17$, but for the *other* acute angle $\cos = 8/17$.

9. Complementary angle identity

Strategy. Sine and cosine of complementary angles are equal: $\sin(x^\circ) = \cos(90^\circ - x^\circ)$. So if $\cos A = \sin B$, then $A + B = 90^\circ$.

Problem. If $\cos(4x - 10)^\circ = \sin(2x + 40)^\circ$, find x.

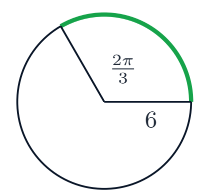
Solution. The two angles are complementary: $(4x - 10) + (2x + 40) = 90$, so $6x + 30 = 90$, giving $x = 10$.

Trap. Do not set the two arguments equal to each other. They sum to 90, not match, because one is a sine and the other a cosine.

SAT move. Unsure? Plug each answer choice in for x; the correct one makes the cosine and the sine equal.

10. Radians and arc length

Strategy. π radians $= 180^\circ$, so convert with degrees $\times \frac{\pi}{180}$. In radians the formulas simplify: arc length $= r\theta$ and sector area $= \frac{1}{2}r^2\theta$, with no $\frac{\theta}{360}$ factor.



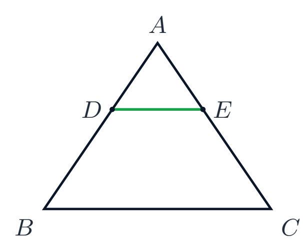
Problem. A central angle of $\frac{2\pi}{3}$ radians is drawn in a circle of radius 6. Find the length of the arc it cuts off.

Solution. In radians, arc $= r\theta = 6 \cdot \frac{2\pi}{3} = 4\pi$. As a check, $\frac{2\pi}{3} = 120^\circ$, and $\frac{120}{360} \cdot 2\pi(6) = 4\pi$ agrees.

Trap. In radians, use $r\theta$ directly; do not *also* divide by 360. Only the degree formula carries the $\frac{\theta}{360}$.

11. Similar triangles and proportions

Strategy. A line parallel to one side of a triangle cuts the other two sides in the same ratio and creates a similar triangle. Set up a proportion of corresponding sides.



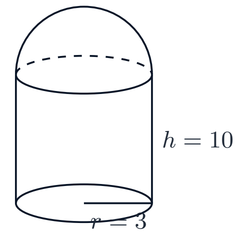
Problem. In triangle ABC, DE is parallel to BC with D on AB, E on AC. If $AD = 4$, $DB = 6$ and $DE = 8$, find BC.

Solution. Triangles ADE and ABC are similar, so $\frac{AD}{AB} = \frac{DE}{BC}$. With $AB = AD + DB = 10$: $\frac{4}{10} = \frac{8}{BC}$, so $BC = 20$.

Trap. Use the full side $AB = 10$ in the ratio, not $AD = 4$. And a ratio of areas is the *square* of the ratio of sides.

12. Composite solids and volume

Strategy. Split a composite solid into standard pieces and add their volumes. The SAT gives you the volume formulas, so the real skill is choosing the right pieces and matching their dimensions.
Problem. A silo is a cylinder of radius 3 and height 10 topped by a hemisphere of radius 3. Find its total volume, in terms of π.
Solution. Cylinder $= \pi r^2 h = \pi(9)(10) = 90\pi$. Hemisphere $= \frac{1}{2} \cdot \frac{4}{3}\pi r^3 = \frac{2}{3}\pi(27) = 18\pi$. Total $= 90\pi + 18\pi = 108\pi$.
Trap. A hemisphere is *half* a sphere, $\frac{2}{3}\pi r^3$, not $\frac{4}{3}\pi r^3$. And both pieces must share the same radius.

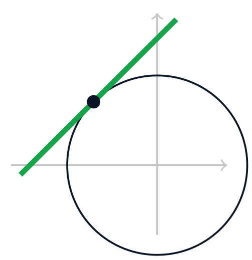


Part 4. Putting it together

four hard questions, worked in full

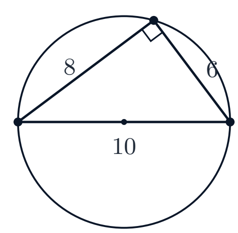
A. A tangent pinned down by the discriminant

Problem. The line $y = x + k$ is tangent to the circle $x^2 + y^2 = 8$. Find the positive value of k.
Solution. Substitute $y = x + k$: $x^2 + (x+k)^2 = 8 \Rightarrow 2x^2 + 2kx + (k^2 - 8) = 0$. Tangent means exactly one solution, so the discriminant is zero: $(2k)^2 - 4(2)(k^2 - 8) = 0 \Rightarrow -4k^2 + 64 = 0 \Rightarrow k^2 = 16$. The positive value is $k = 4$.
Combines. the circle equation (3) and line-meets-curve (6), with the discriminant of a quadratic.



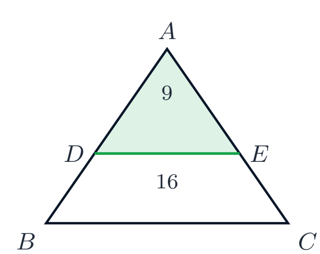
B. A right triangle hiding in a circle

Problem. A right triangle is inscribed in a circle so that its hypotenuse is a diameter. Its legs are 6 and 8. Find the area of the circle.
Solution. An angle inscribed in a semicircle is a right angle, so the hypotenuse is the diameter. By the Pythagorean theorem the hypotenuse is $\sqrt{6^2 + 8^2} = 10$, so the radius is 5 and the area is $\pi(5)^2 = 25\pi$.
Combines. the inscribed-angle theorem (1), the Pythagorean theorem, and circle area.



C. An area ratio through a parallel line

Problem. In triangle ABC, DE is parallel to BC. The area of triangle ADE is 9 and the area of trapezoid $DBCE$ is 16. Find $AD : AB$.
Solution. The whole triangle ABC has area $9 + 16 = 25$. Triangles ADE and ABC are similar, and a ratio of areas is the *square* of the ratio of sides: $\left(\frac{AD}{AB}\right)^2 = \frac{9}{25}$, so $\frac{AD}{AB} = \frac{3}{5}$.
Combines. similar triangles (11) with the area-ratio rule.

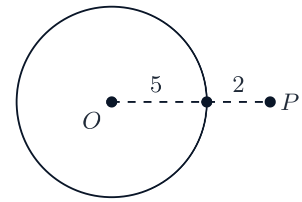


D. The nearest point on a circle

Problem. Find the shortest distance from the point $(7,1)$ to the circle $(x-2)^2+(y-1)^2=9$.

Solution. The centre is $(2,1)$ and the radius is 3. The distance from the point to the centre is $\sqrt{(7-2)^2+(1-1)^2}=5$. The nearest point on the circle lies on that line, so the shortest distance is $5-3=2$.

Combines. the distance formula (5) with the circle equation (3). The trap: subtract the radius.



On test day. Draw the figure if none is given, and mark every right angle. When a circle appears, look for an inscribed angle or a tangent meeting a radius. On a coordinate grid, reach for Desmos to graph and confirm.

Formula quick reference

♦ given on the SAT reference sheet; everything else, you must memorise.

Circles

- Inscribed angle $=\frac{1}{2}$ central (same arc)
- Arc $=\frac{\theta}{360}2\pi r = r\theta$ (radians)
- Sector $=\frac{\theta}{360}\pi r^2=\frac{1}{2}r^2\theta$
- Circle: $(x-h)^2+(y-k)^2=r^2$
- Tangent $\perp$ radius at contact

Coordinate geometry

- Distance $=\sqrt{(\Delta x)^2+(\Delta y)^2}$
- Midpoint $=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$
- Slope $m=\frac{\Delta y}{\Delta x}$
- Parallel: equal m. Perpendicular: $m_1m_2=-1$

Triangles and trigonometry

- ♦ $45\text{-}45\text{-}90$: $1:1:\sqrt{2}$
- ♦ $30\text{-}60\text{-}90$: $1:\sqrt{3}:2$
- SOH CAH TOA
- $\sin x=\cos(90^\circ - x)$
- π rad $=180^\circ$
- Area ratio $=$ (side ratio)^2

Solids

- ♦ Cylinder $V=\pi r^2 h$
- ♦ Cone $V=\frac{1}{3}\pi r^2 h$
- ♦ Sphere $V=\frac{4}{3}\pi r^3$
- Composite: add or subtract the pieces